How are differential equations used to model and analyze the response of electrical filters

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Extended Essay: HL Math AA

This study explores the application of ordinary differential equations (ODEs) in modelling and analyzing the behavior of electrical filters, with a focus on low-pass filters. By leveraging ODEs, the research aims to provide a deeper understanding of how these filters respond to various inputs, highlighting their mathematical properties and functional characteristics.

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“From a drop of water, a logician could infer the possibility of an Atlantic or a Niagara without having seen or heard of one or the other...” – Sir Arthur Conan Doyle [1]

Contents

1. Motivation

Circuits of unparalleled complexity operate as an integral part of our daily lives. The flexibility and diversity of these ubiquitous circuits stimulated curiosity in me from a young age.

This curiosity only grew over the years, and so I decided to learn how to make printed circuit boards (PCBs) which are used in all electronics. However, while designing specific circuits such as power regulation, or digital filters, I needed to look up on a website the component values in order to reproduce the circuit behaviour I desired. [2]

Specifically, I wanted to derive and fundamentally understand the equation provided for a low-pass filter, which I used in the design of a microphone. A low-pass filter was used to filter out frequencies higher than 20000 Hz, as the human ear cannot hear at a higher frequency than that, so it is useless to record higher frequencies.

This led me to dive deeper into how these seemingly arbitrary component values are derived for specific filter behaviours. After researching and outlining the strong prevalence of math in this discipline, I finalized my research question:

"How are differential equations used to model and analyze the response of electrical filters?"

This question reflects my desire to understand the mathematical foundations underlying circuit design, especially in filters.

2. Electrical Circuits

The history of electrical circuits dates back to the early 19th century and is the foundation of modern technology, utilized in **everything** from household appliances to complex industrial systems.

An electrical circuit is a *closed loop* consisting of **passive** components through which electrons can flow.

Closed-loop refers to the existence of a continuous path that allows charge to flow from one end back to the other end, forming a complete loop.

2.0.1. Quantities

The fundamental quantities in electrical circuits are **current** and **voltage**. Current is the rate of flow of electrons, measured in *amperes* (A), whereas Voltage is the force that drives the current, measured in *volts* (V).

In addition to current and voltage, the following are key electrical quantities that will be mentioned.

Quantity and Symbol	Units	Description
Voltage (v)	Volt (V)	Potential difference between two points
Charge (Q)	Coulomb (C)	Amount of electric charge
Flux (φ)	Weber (Wb)	Time integral of voltage
Current (i)	Ampere (A)	Flow of electric charge
Resistance (R)	Ohm (Ω)	Opposition to a flow of current
Inductance (L)	Henry (H)	Opposition to a change in current
Capacitance (C)	Farad (F)	Opposition to a change in voltage

Table 1: Electrical Quantities

2.0.2. Current types

Electrical circuits can run on either direct current (DC) or alternating current (AC). In DC, electrons flow consistently in one direction only. Whereas in AC, the flow of electrons periodically switches direction, thus alternating. [3]

So, AC voltage can be characterized by its sinusoidal pattern. The frequency of this oscillation, typically 50 or 60 Hz in most countries [4], determines how many times per second the current changes direction. This nature allows us to model AC voltage using the **sine** function.

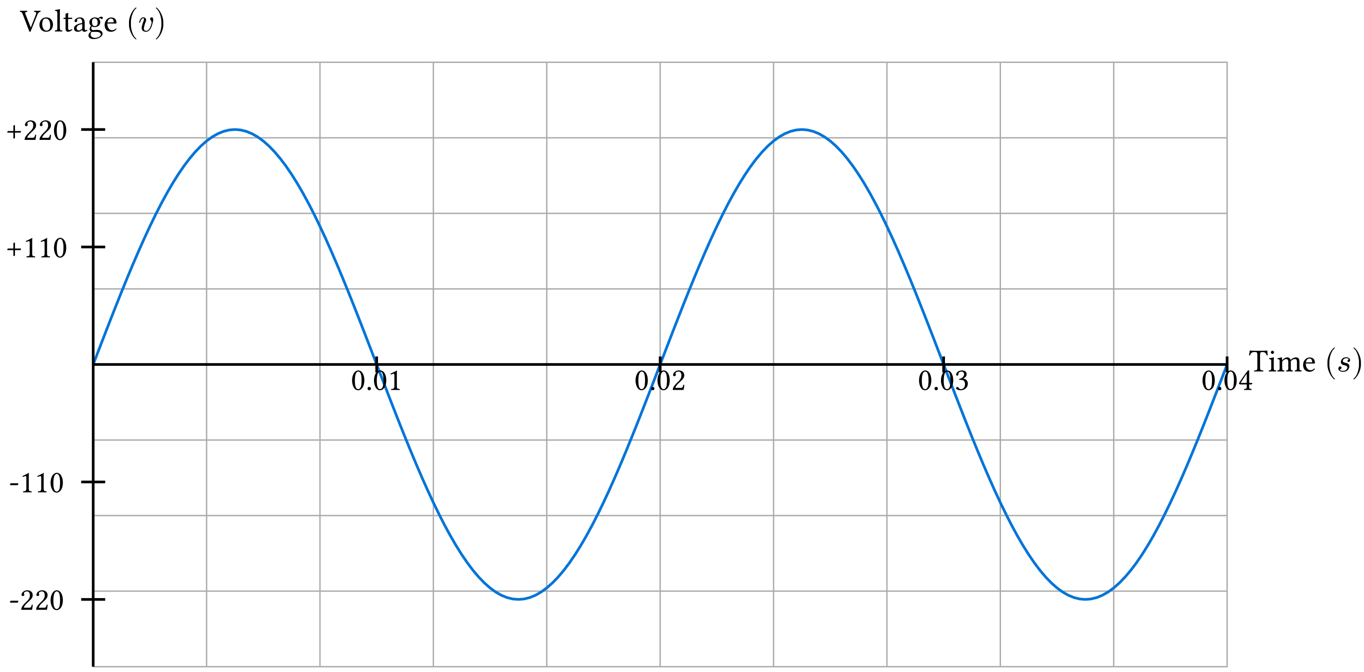


Figure 1: Function of an AC wave at $220v$ and 50Hz

$$v(t) = 220\sin(100\pi t) \tag{1}$$

This equation takes the general form

$$A\sin(\omega t - \varphi) \tag{2}$$

Where A is the maximum amplitude of the wave.

Angular frequency, denoted as ω, is related to the regular frequency f by the equation.

$$\omega = 2\pi f \tag{3}$$

Here, f is the frequency in hertz (Hz), which measures the number of oscillations per second. The factor 2π converts this into radians per second, which is useful for describing oscillations in terms of angles. This is because one complete cycle of a wave corresponds to 2π radians.

Lastly, φ represents the phase shift, how many units on the time axis the wave has been shifted.

2.0.3. Passive components

Electrical filters, being electrical circuits fundamentally, only use the following 3 passive components. [5]

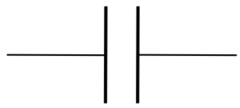
Component and Symbol	Unit	Description	Representation
Resistor R	Ohms (Ω)	Opposes the flow of current	
Inductor L	Henry (H)	Stores energy in a magnetic field	
Capacitor C	Farad (F)	Stores energy in an electric field	

Table 2: Passive Components

2.0.4. Electrical Systems

Since current (i) describes the rate of flow of charge (Q), we can express it as a time differential. [6]

$$i = \frac{\mathrm{d}Q}{\mathrm{d}t} \tag{4}$$

Similarly, as described by Faraday's law of induction, voltage (v) in a circuit can also be expressed as a time differential of flux (φ) [7].

$$v = \frac{d\Phi}{\mathrm{d}t} \tag{5}$$

And finally, Ohm's law describes resistance (R) as the constant of proportionality between current and voltage.

$$v = iR \tag{6}$$

These 3 equations allow us to define a system of equations that describes the mathematical relationship between all of our electrical circuit quantities. [8]

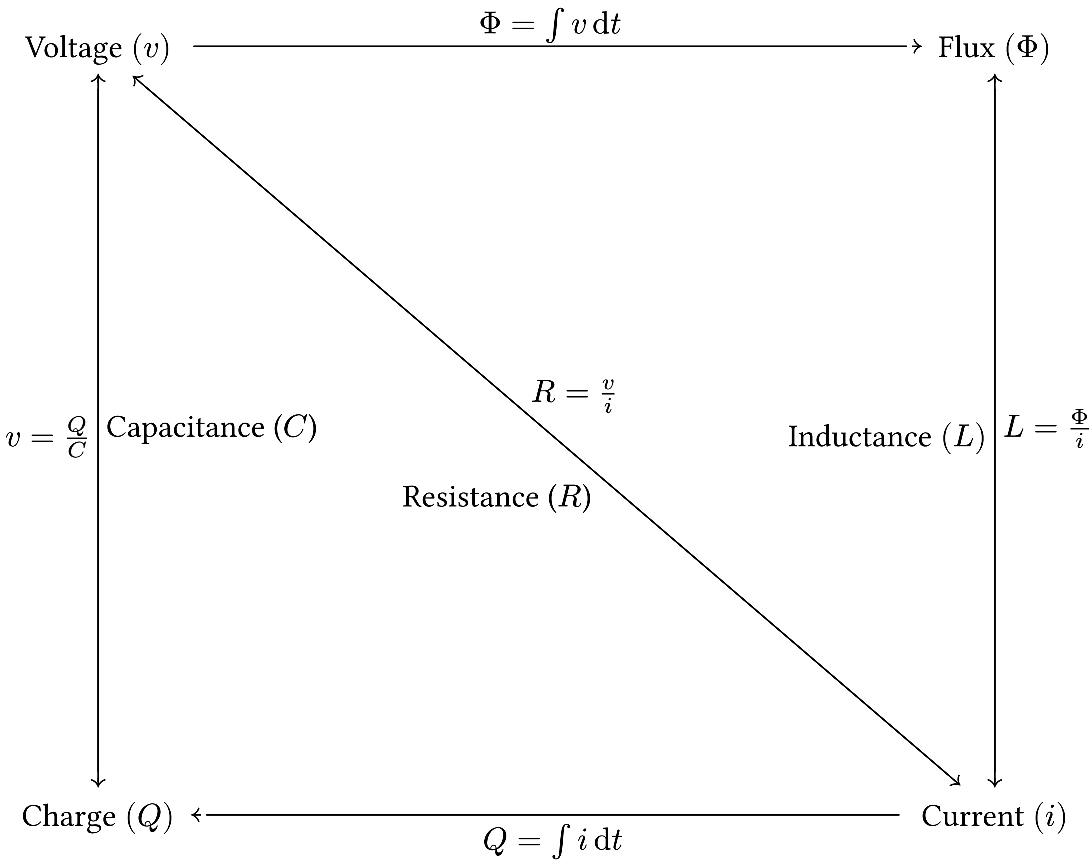


Figure 2: The Electrical System

Similar to how resistance is the constant of proportionality between voltage and current, we can describe capacitance and inductance as constants of proportionality between charge and voltage, and flux and current respectively.

This system of equations allows us to simplify all quantities present in our electrical circuit into any single variable of this system.

This will enable us to model any electrical circuit and specifically electrical filters as an ordinary differential equation.

3. Ordinary Differential Equations

A differential equation equates the values of any function to the values of the function's derivatives. [9]

A differential equation is called an ordinary differential equation (ODE) if the function is bounded by a single variable. For example:

$$v''(t) + 5v'(t) + v(t) = 0 \tag{7}$$

An n-order ODE will have up to the $\frac{d^n y}{dx^n}$ derivative.

An n-order ODE is linear if it can be written in the form:

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = r(x) \tag{8}$$

Furthermore, if $r(x) = 0$, the ODE is considered homogeneous; otherwise non-homogeneous.

3.1. Applications

Differential equations, under the branch of calculus, can describe rates of change in various phenomena. It has various applications in physics, economics, biology, and electrical engineering.

This is due to their inherent strength in modelling and analyzing systems of change, thereby allowing us to make predictions about their behavior and evolution over time.

3.2. Solving ODEs

Solving ordinary differential equations (ODEs) can be approached through numerical and analytical methods, each offering unique advantages. Let's explore them for the following example.

$$2\frac{d^2 v}{dt^2} + 3\frac{dv}{dt} + 7v = 0 \tag{9}$$

3.2.1. Numerical Methods

Numerical methods, such as **Euler's method** [10], provide approximate solutions and are particularly useful when an exact analytical solution is difficult or impossible to find.

Euler's method uses small steps as a representation of dx to approximate the solution to an ODE for a specific input. This will not lead to an exact solution, however the smaller the step of dx that we use, the closer we will get.

Likewise, **MATLAB** is a powerful software tool that can be used to numerically solve ODEs due to the power of electrical computing. [11]

Advantages	Limitations
Can handle complex, non-linear ODEs	Solutions are approximations and may contain errors

Advantages	**Limitations**
Suitable for systems with varying parameters	Computationally expensive for large systems

Numerically solving Equation 9 using **Euler's method**:

Let,

$$q = \frac{\mathrm{d}v}{\mathrm{d}t} \tag{10.1}$$

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\mathrm{d}v^2}{\mathrm{d}^2t} \tag{10.2}$$

We can solve for $\frac{\mathrm{d}v^2}{\mathrm{d}^2t}$ in Equation 9

$$2\frac{\mathrm{d}v^2}{\mathrm{d}^2t} + 3\frac{\mathrm{d}v}{\mathrm{d}t} + 7v = 0 \tag{11.1}$$

$$\frac{\mathrm{d}v^2}{\mathrm{d}^2t} = -\frac{3}{2}\frac{\mathrm{d}v}{\mathrm{d}t} - \frac{7}{2}v \tag{11.2}$$

Substituting this into Equation 10

$$\frac{\mathrm{d}q}{\mathrm{d}t} = -\frac{3}{2}\frac{\mathrm{d}v}{\mathrm{d}t} - \frac{7}{2}v \tag{12.1}$$

$$= -\frac{3}{2}q - \frac{7}{2}v \tag{12.2}$$

Now we can choose a constant small time step, Δt, and start with initial values for v and q at $t = 0$.

Iterate over a range of time steps:

$$v_{n+1} = v_n + \Delta t \cdot q_n \tag{13}$$

$$q_{n+1} = q_n + \Delta t \cdot \left(-\frac{3}{2}q_n - \frac{7}{2}v_n\right) \tag{14}$$

Until the desired time range is reached.

Or by using **MATLAB**:

MATLAB

```matlab
% Define the differential equation
ode = @(t, y) [y(2); -3/2 * y(2) - 7/2 * y(1)];

% Solve the ODE for initial conditions y(0) = 1, y'(0) = 0
[t1, y1] = ode45(ode, [0 10], [1; 0]);

% Solve the ODE for initial conditions y(0) = 0, y'(0) = 1
[t2, y2] = ode45(ode, [0 10], [0; 1]);

% Plot y vs t for both solutions
figure;
subplot(2, 2, 1);
plot(t1, y1(:, 1), 'b');
title('y(0) = 1, y''(0) = 0');
xlabel('t'); ylabel('y');

subplot(2, 2, 3);
plot(t2, y2(:, 1), 'b');
title('y(0) = 0, y''(0) = 1');
xlabel('t'); ylabel('y');

% Plot phase portraits y' vs y for both solutions
subplot(2, 2, 2);
plot(y1(:, 1), y1(:, 2), 'b');
title('y(0) = 1, y''(0) = 0');
xlabel('y'); ylabel('y''');

subplot(2, 2, 4);
plot(y2(:, 1), y2(:, 2), 'b');
title('y(0) = 0, y''(0) = 1');
xlabel('y'); ylabel('y''');
```

Listing 1: MATLAB Script to solve Equation 9 using ode45

Which produces two solutions:

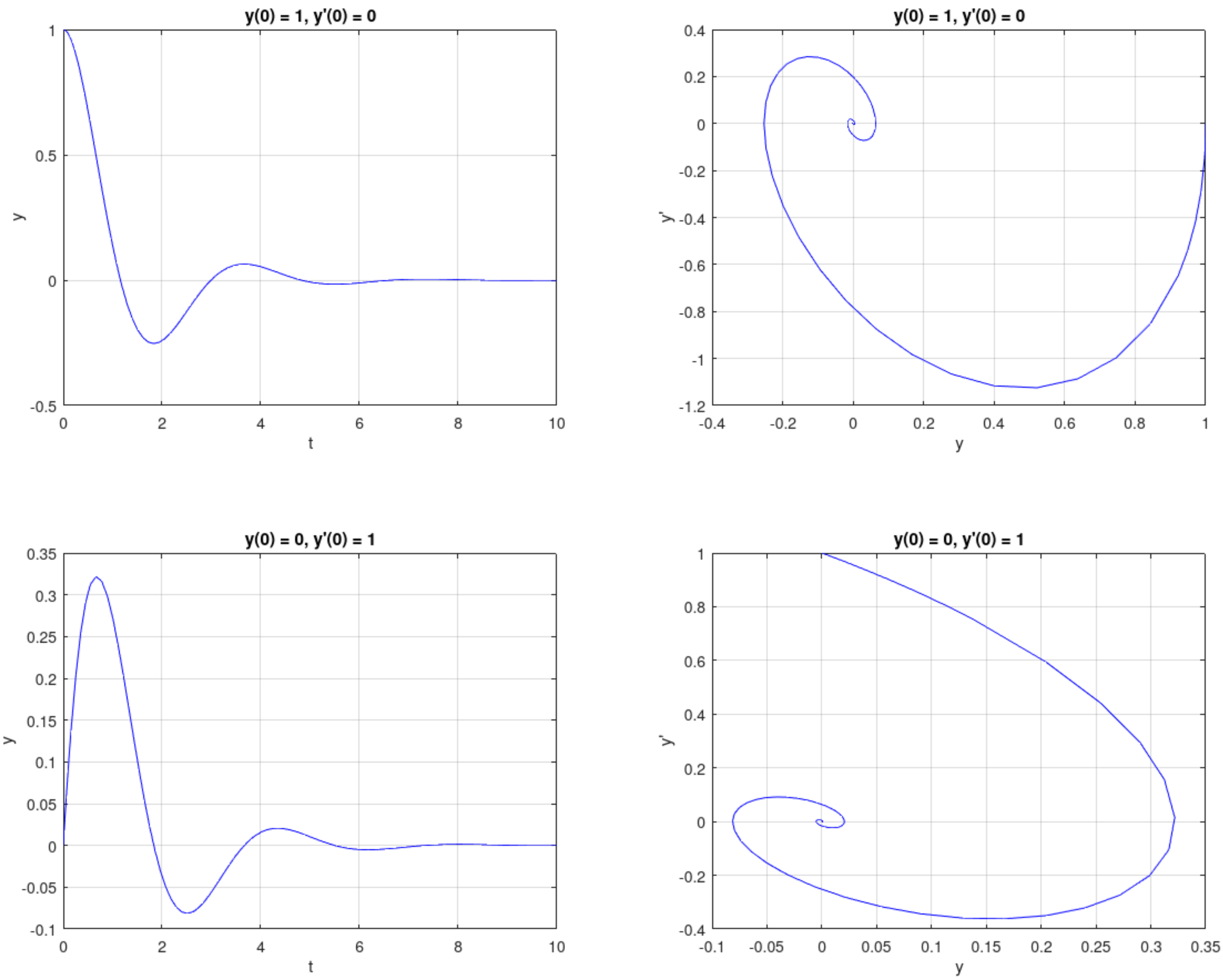


Figure 3: Approximate solution of $v(t)$ [11]

The phase plot presents how the derivative of v changes as v changes, always approaching $(0, 0)$.

Note the relatively jagged lines, especially in the phase plots. This is due to the numerical solution used not providing precise enough values.

3.2.2. Analytical Solutions

Analytical solutions provide exact, closed-form expressions. These methods give insight into the behavior and response of the system and allow for precise tuning of parameters. [12]

Advantages	Limitations
Provide exact solutions.	Not always possible for complex ODEs.
Offer a deeper understanding of system dynamics.	Often requires simplifying assumptions.

To analytically solve any ODE, we must realize that a special property of homogeneous ODEs are that their derivatives are proportional to each other. This is due to the fact that they are homogeneous.

The only typical function that also lends itself to this property is $y = e^{rx}$. Where its derivatives are proportional to itself.

$$y = e^{rx} \tag{15.1}$$

$$y' = re^{rx} \tag{15.2}$$

$$y'' = r^2 e^{rx} \tag{15.3}$$

$$\ldots \tag{15.4}$$

3.2.2.1. General Solution of a first-order homogeneous ODE

If we take a general first-order homogeneous ODE $ay' + by = 0$ and let the solution take the form $y = Ce^{rx}$.

C is included as a starting condition and is determined through an initial y and x value.

Substituting the solution and its derivative into the ODE.

$$a(rCe^{rx}) + b(Ce^{rx}) = 0 \tag{16}$$

We can further factor Ce^{rx} out and get a simple algebraic expression in terms of r.

$$ar + b = 0 \tag{17}$$

This is referred to as the **characteristic equation**, the root of this equation has a singular solution, $r = -\frac{a}{b}$, and so.

$$y = Ce^{-\frac{a}{b}x} \tag{18}$$

3.2.2.2. General Solution of a second-order homogeneous ODE

For a second-order homogeneous ODE, we can assume a similar **characteristic equation** of a greater order:

$$ar^2 + br + c = 0 \tag{19}$$

However, as you may have already guessed, r may have multiple solutions, while the initial solution assumption of $y = e^{rx}$ only expects there to be one possible value for r.

This is because the initial function and its second derivative do not necessarily have a constant of proportionality.

However, since the ODE is linear or homogeneous, the solution can also be described **linearly**, so we can assume the solution to be:

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x} \tag{20}$$

Here we have $C_1 e^{r_1 x}$ representing the relationship between the function and its first derivative, and $C_2 e^{r_2 x}$ representing the relationship between the first derivative and the second derivative.

Now we can break it into cases depending on the determinant of the characteristic equation.

The first case of distinct real roots is already described perfectly by the assumed solution.

The second case of repeated real roots can be simply rewritten as:

$$y = C_1 e^{rx} + C_2 x e^{rx} \tag{21}$$

The third case of complex conjugate roots has roots:

$$r = \alpha \pm \beta i \tag{22}$$

And so the solution would be [13]

$$y = C_1 e^{\alpha + \beta i x} + C_2 e^{\alpha - \beta i x} \tag{23}$$

However, this is an undesirable form as it's not easily graphable or useful. Luckily, it is rather similar to the left-hand side of Euler's Formula, which claims that [14]

$$e^{ix} = \cos(x) + i \sin(x) \tag{24}$$

Proof using Maclaurin Series Expansions:

The series expansions of the concerned functions are: [15]

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... \tag{25.1}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... \tag{25.2}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... \tag{25.3}$$

We can extend the e^x expansion for e^{ix}.

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + ... \tag{26}$$

This can be simplified, given that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, and so on:

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + ... \tag{27}$$

Now we can separate the real and imaginary parts.

$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...\right) \tag{28}$$

By separating the real and imaginary parts, it's revealed that the real expansion is $\cos(x)$ and the imaginary expansion is $\sin(x)$

$$e^{ix} = \cos(x) + i \sin(x) \blacksquare \tag{29}$$

Now we can rewrite the solution.

$$y = C_1 e^{(\alpha + \beta i)x} + C_2 e^{(\alpha - \beta i)x} \tag{30.1}$$

$$= C_1 e^{\alpha x} e^{\beta i x} + C_2 e^{\alpha x} e^{-\beta i x} \tag{30.2}$$

$$= e^{\alpha x}\left(C_1 e^{\beta i x} + C_2 e^{-\beta i x}\right) \quad (30.3)$$

$$= e^{\alpha x}(C_1(\cos(\beta x) + i\sin(\beta x)) + C_2(\cos(\beta x) - i\sin(\beta x))) \quad (30.4)$$

$$= e^{\alpha x}((C_1 + C_2)\cos(\beta x) + i(C_1 - C_2)\sin(\beta x)) \quad (30.5)$$

Simply put,

$$A = C_1 + C_2 \quad (31.1)$$

$$B = i(C_1 - C_2) \quad (31.2)$$

$$y = e^{\alpha x}(A\cos(\beta x) + B\sin(\beta x)) \quad (31.3)$$

Now we can attempt to analytically solve Equation 9

$$2\frac{\mathrm{d}^2 v}{\mathrm{d}t^2} + 3\frac{\mathrm{d}v}{\mathrm{d}t} + 7v = 0 \quad (32)$$

Which has the characteristic equation of

$$2r^2 + 3r + 7 = 0 \quad (33.1)$$

$$r = \frac{-3 \pm \sqrt{3^2 - 4 \cdot 2 \cdot 7}}{2 \cdot 2} \quad (33.2)$$

$$r = -\frac{3}{4} \pm i\left(\frac{\sqrt{47}}{4}\right) \quad (33.3)$$

Since the roots are complex, we can assume the general solution we derived earlier.

$$v(t) = e^{-\frac{3}{4}t}\left(A\cos\left(\frac{\sqrt{47}}{4}t\right) + B\sin\left(\frac{\sqrt{47}}{4}t\right)\right) \quad (34)$$

Taking the same initial conditions $(v(0) = 0, v'(0) = 1)$ as our first analytical solution.

$$v(0) = 0 = e^0(A \cdot 1 + b \cdot 0) \quad (35.1)$$

$$A = 0 \quad (35.2)$$

$$v'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\frac{3}{4}t}\left(A\cos\left(\frac{\sqrt{47}}{4}t\right) + B\sin\left(\frac{\sqrt{47}}{4}t\right)\right)\right) \quad (36.1)$$

$$v'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\frac{3}{4}t} \cdot B\sin\left(\frac{\sqrt{47}}{4}t\right)\right) \quad (36.2)$$

$$v'(t) = -\frac{3}{4}e^{-\frac{3}{4}t} \cdot B\sin\left(\frac{\sqrt{47}}{4}t\right) + e^{-\frac{3}{4}t} \cdot B \cdot \frac{\sqrt{47}}{4}\cos\left(\frac{\sqrt{47}}{4}t\right) \quad (36.3)$$

$$v'(0) = 1 = B \cdot \frac{\sqrt{47}}{4} \quad (36.4)$$

$$B = \frac{4}{\sqrt{47}} \quad (36.5)$$

So our solution will be

$$v(t) = e^{-\frac{3}{4}t} \cdot \frac{4}{\sqrt{47}} \sin\left(\frac{\sqrt{47}}{4}t\right) \tag{37}$$

By graphing this solution, we can verify that our analytical solution is equivalent to our numerical solution.

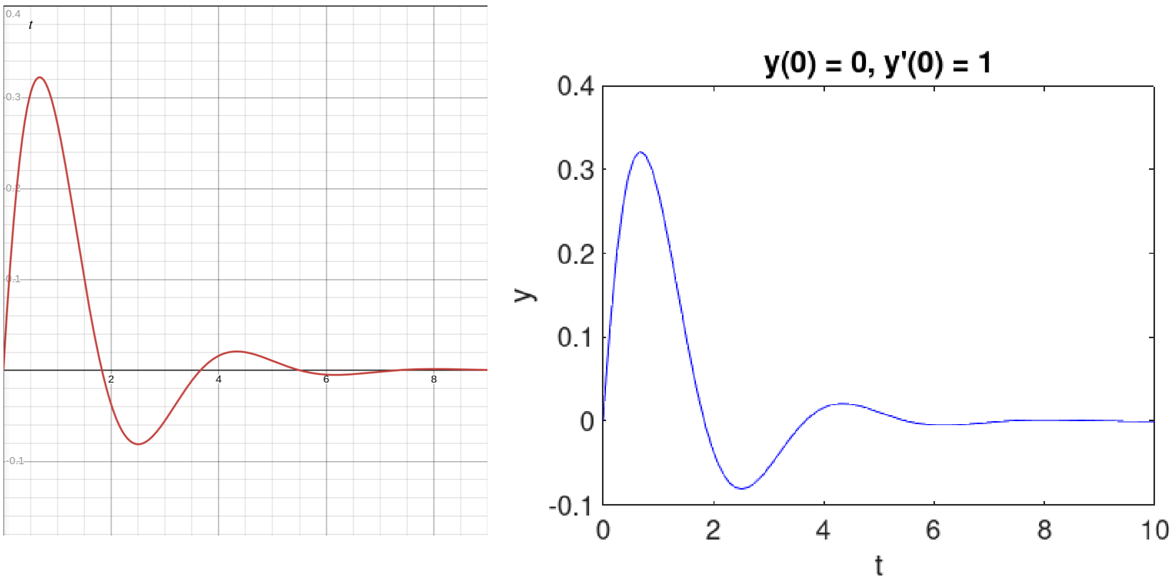


Figure 4: Graph of the analytical solution for v(t)

4. Modelling

There are two ways we can model the sinusoidal voltage and current.

The most common method uses trigonometric functions.

$$v(t) = V_0 \cos(\omega t) \tag{38}$$

where V_0 is the amplitude, ω is the angular frequency.

However, we can also use Euler's formula that we explored earlier as a form for sinusoidal functions. [16]

$$e^{j\omega t} = \cos(\omega t) + j \sin(\omega t) \tag{39}$$

Note that j will be used to represent the imaginary number $\sqrt{-1}$ instead of i to avoid confusion with current.

This form of sinusoidal functions makes it easier to differentiate and integrate. It further simplifies calculations involving addition, subtraction, and multiplication of sinusoids, especially when dealing with phase shifts.

We can also verify that both forms are equivalent in the real domain. [17]

$$v(t) = \Re\{V_0 e^{j\omega t}\} = V_0 \cos(\omega t) \tag{40}$$

So we can write our alternating current and voltage waves in Euler's form.

$$\begin{aligned} i(t) &= i_0 e^{j\omega t} &= i_0 \cos(\omega t) \\ v(t) &= v_0 e^{j\omega t} &= v_0 \cos(\omega t) \end{aligned} \tag{41}$$

4.0.1. Components

Impedance and reactance describe the behavior of passive components in AC circuits. [18]

Impedance (Z) is a complex number that combines **resistance** and **reactance**, representing how a component opposes the flow of alternating current (AC). It is expressed as:

$$Z = R + jX \tag{42}$$

- **Resistance** (R) is the real part, opposing both AC and DC current, measured in ohms (ω).
- **Reactance** (X) is the imaginary part, opposing changes in current and causing phase shifts between voltage and

4.0.1.1. Reactance in Components

Reactance varies with angular frequency ω and is categorized into:

- **Inductive Reactance** $X_L = \omega L$
  - Increases with frequency.
  - Causes current to lag behind voltage.
- **Capacitive Reactance** $X_C = -\frac{1}{\omega C}$
  - Decreases with frequency.
  - Causes current to lead voltage.

4.0.1.2. Impedance in Components

1. **Resistors**:
   - Reactance $X = 0$
   - Impedance: $Z_R = R$
   - Voltage drop: $V_R = iR$
2. **Inductors**:
   - Resistance $R = 0$
   - Impedance: $Z_L = j\omega L$
   - Voltage drop: $V_L = i \cdot j\omega L$
3. **Capacitors**:
   - Resistance $R = 0$
   - Impedance: $Z_C = \frac{1}{j\omega C}$
   - Voltage drop: $V_C = i \cdot \frac{1}{j\omega C}$

Referring to Figure 2, we are also able to represent the voltage drop of inductors and capacitors as the derivative and integral of current.

Since we would like to describe them homogeneously in terms of voltage and current, we need to rearrange the equations.

For inductors.

$$\begin{aligned} L &= \frac{\Phi}{i} \\ \Phi &= Li \\ \frac{\mathrm{d}}{\mathrm{d}t}(\Phi) &= \frac{\mathrm{d}}{\mathrm{d}t}(Li) \\ v &= L\frac{\mathrm{d}i}{\mathrm{d}t} \end{aligned} \tag{43}$$

Thus, we can define,

$$V_L = \frac{\mathrm{d}i}{\mathrm{d}t}L \equiv i(j\omega L) \tag{44}$$

Proof:

$$\begin{aligned} \frac{\mathrm{d}i}{\mathrm{d}t}L &= \frac{\mathrm{d}}{\mathrm{d}t}(i(t))L \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(i_0 e^{j\omega t})L \\ &= i_0 e^{j\omega t} \cdot (j\omega)L \\ &= i \cdot j\omega L \qquad \blacksquare \end{aligned} \tag{45}$$

For capacitors.

$$v = \frac{Q}{C} \\ = \frac{1}{C} \int i \,\mathrm{d}t \tag{46}$$

Thus,

$$V_C = \frac{1}{C} \int i \,\mathrm{d}t \equiv i \cdot \frac{1}{j\omega C} \tag{47}$$

Proof:

$$\frac{1}{C} \int i \,\mathrm{d}t = \frac{1}{C} \int \left(i_0 e^{j\omega t}\right) \mathrm{d}t \\ = \frac{1}{C} \cdot \frac{1}{j\omega} \left(i_0 e^{j\omega t}\right) \\ = \frac{1}{j\omega C} \cdot i \quad \blacksquare \tag{48}$$

Since all of the waves will oscillate around $y = 0$, there is no integration constant C.

4.0.2. Circuits

Now that we can model individual components, we can make use of **Kirchhoff's Voltage Laws** to model a full circuit.

Kirchhoff's voltage law (KVL) states that the total sum of all voltage drops in a closed-loop circuit is equal to zero [19]

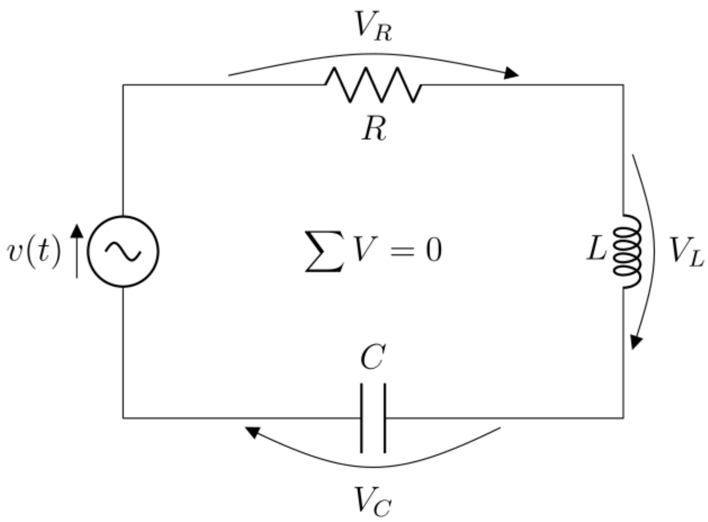


Figure 5: Simple RLC circuit with a closed loop

So we can derive an equation to model Figure 5

$$\sum v = 0 \\ v(t) - V_R - V_L - V_C = 0 \\ v(t) = V_R + V_L + V_C \tag{49}$$

Where:

- $V_R = i(t) \cdot R$
- $V_L = i \cdot j\omega L \equiv \frac{\mathrm{d}i(t)}{\mathrm{d}t} L$ (Equation 44)
- $V_C = i \cdot \frac{1}{j\omega C} \equiv \frac{1}{C} \int i \, \mathrm{d}t$ (Equation 46)

Remember that R, L and C are constant component values.

So

$$v(t) = i(t) \cdot R + \frac{\mathrm{d}i(t)}{\mathrm{d}t} L + \frac{1}{C} \int i \, \mathrm{d}t \tag{50}$$

Differentiating both sides to eliminate the integral.

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = \frac{i(t)}{C} + R \frac{\mathrm{d}i(t)}{\mathrm{d}t} + L \frac{\mathrm{d}^2 i(t)}{\mathrm{d}t^2} \tag{51}$$

This is our second-order linear ordinary differential equation that describes the circuit Figure 5

Furthermore, if we assume the initial voltage to be in a steady state, or $\frac{\mathrm{d}v(t)}{\mathrm{d}t} = 0$.

We can solve it as we did for Equation 9

For example, if

$$\begin{aligned} L &= 2H \\ R &= 3\Omega \\ C &= 7F \end{aligned} \tag{52}$$

Then the solution of Equation 51 which models Figure 5 would have the exact same solution as Figure 4 considering the same initial conditions: $\frac{\mathrm{d}v(t)}{\mathrm{d}t} = 0$, $\frac{\mathrm{d}^2 v(t)}{\mathrm{d}t^2} = 1$.

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = e^{-\frac{3}{4}t} \cdot \frac{4}{\sqrt{47}} \sin\left(\frac{\sqrt{47}}{4} t\right) \tag{53}$$

Now, with all the necessary tools, we can model and analyze the response of a low-pass filter.

5. The Lowpass Filter

RLC circuits can be used to form filters, the most common of which includes a low pass filter used in microphones, which looks like the following

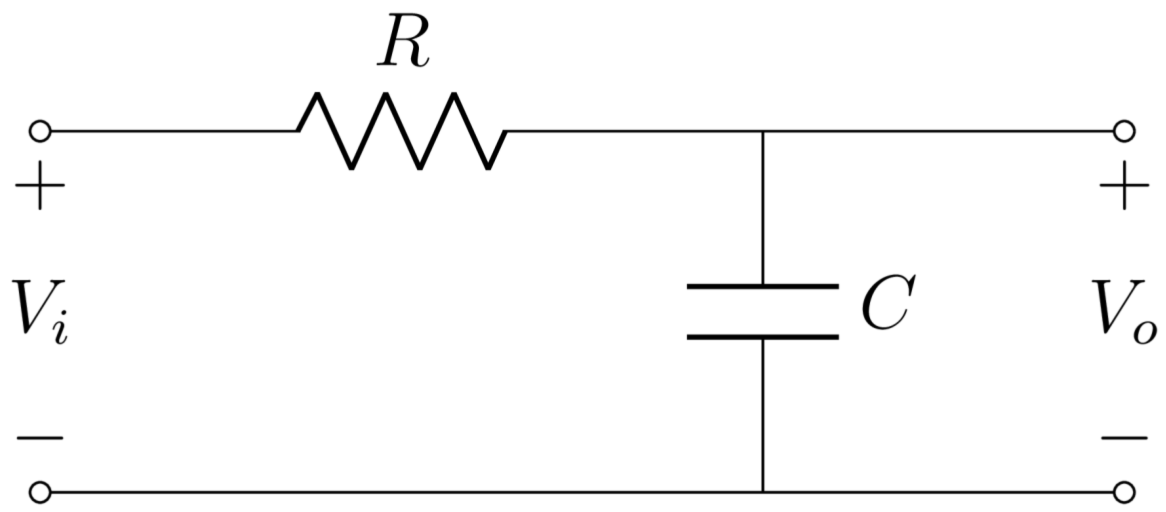


Figure 6: Low-Pass filter

This filter lowers the voltage of frequencies that are outside a certain range of frequencies.

By analyzing this circuit mathematically, not only can we determine the cutoff frequency given the RLC component values, but we can also determine component values given the cutoff frequency we wish to achieve.

However, note that this circuit is in parallel, and there are multiple loops compared to our previous model. This means we will have to model it slightly differently. [20]

5.1. Model

Using KVL on the loop of the circuit encompassing input voltage, the resistor, and output voltage, we can deduce that.

$$\begin{aligned} V_i - V_R - V_o &= 0 \\ V_R &= V_i - V_o \\ i \cdot R &= V_i - V_o \end{aligned} \tag{54}$$

To determine the current across the capacitor.

$$\begin{aligned} V_o &= \frac{Q}{C} \\ V_o C &= Q \\ \frac{\mathrm{d}}{\mathrm{d}t}(V_o C) &= \frac{\mathrm{d}}{\mathrm{d}t}(Q) \\ C \frac{\mathrm{d}V_o}{\mathrm{d}t} &= i \end{aligned} \tag{55}$$

And so,

$$V_i - V_o = i \cdot R = RC\frac{\mathrm{d}V_o}{\mathrm{d}t} \tag{56}$$

Which written as a first order ODE, as there are only two components

$$RC\frac{\mathrm{d}V_o}{\mathrm{d}t} + V_o = V_i \tag{57}$$

5.2. Solutions

The solution goal is to find $V_o(t)$, the output voltage against time, given an input voltage $V_i(t)$

However, since this is a homogeneous ODE model, we have to resolve the differential equation as a sum of solutions.

$$V_o(t) = V_{\mathrm{oh}}(t) + V_{\mathrm{op}}(t) \tag{58}$$

Where $V_{\mathrm{oh}}(t)$ is the homogeneous solution and $V_{\mathrm{op}}(t)$ is the particular solution for $V_o(t)$.

5.2.1. The Homogeneous Solution

The homogeneous equation is obtained by setting the input, $V_i(t)$ to 0:

$$RC\frac{\mathrm{d}V_{\mathrm{oh}}}{\mathrm{d}t} + V_{\mathrm{oh}} = 0 \tag{59}$$

According to Euler, all first order homogeneous solutions can be written in the form

$$V_{\mathrm{oh}}(t) = ce^{kt} \tag{60}$$

Where c is bounded by initial conditions

Substituting Equation 60 into Equation 59 gives the characteristic equation with root k

$$\begin{aligned} RC\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{kt}\right) + e^{kt} &= 0 \\ RCke^{kt} + e^{kt} &= 0 \\ RCk + 1 &= 0 \\ k &= -\frac{1}{RC} \end{aligned} \tag{61}$$

Thus, we can solve the homogeneous equation given the characteristic root k

$$V_{\mathrm{oh}}(t) = ce^{-\frac{t}{RC}} \tag{62}$$

5.2.2. The Particular Solution

For the particular solution, we first assume that the input voltage will follow Equation 41.

$$V_i = V_0 e^{j\omega t} \tag{63}$$

Secondly, we assume that the output voltage will be in Euler's form.

$$\begin{aligned} V_{\mathrm{op}} &= Ae^{j(\omega t+\Phi)} \\ \Rightarrow \frac{\mathrm{d}V_{\mathrm{op}}}{\mathrm{d}t} &= j\omega Ae^{j(\omega t+\Phi)} \end{aligned} \tag{64}$$

Substituting the assumptions into Equation 57.

$$\begin{aligned} RC\left(j\omega Ae^{j(\omega t+\Phi)}\right) + Ae^{j(\omega t+\Phi)} &= V_0 e^{j\omega t} \\ RCj\omega Ae^{j\Phi} + Ae^{j\Phi} &= V_0 \\ Ae^{j\Phi}(RCj\omega + 1) &= V_0 \\ Ae^{j\Phi} &= \frac{V_0}{RCj\omega + 1} \end{aligned} \tag{65}$$

The amplitude A of the output is given by the magnitude of the complex response

$$\begin{aligned} A &= \left|\frac{V_0}{RCj\omega + 1}\right| \\ &= \frac{V_0}{\sqrt{(RC\omega)^2 + 1}} \end{aligned} \tag{66}$$

And the phase Φ is given by the argument or angle of the response.

$$\begin{aligned} \Phi &= \angle V_0 - \angle(j\omega RC + 1) \\ &= -\tan^{-1}(\omega RC) \end{aligned} \tag{67}$$

Hence,

$$V_{\mathrm{op}}(t) = \Re\left\{Ae^{j(\omega t+\Phi)}\right\} = A\cos(\omega t + \Phi) \tag{68}$$

5.2.3. The General Solution

Finally, we can describe the general solution of our model.

$$V_o(t) = ce^{-\frac{t}{RC}} + A\cos(\omega t + \Phi) \tag{69}$$

Where,

$$\begin{aligned} \Phi &= -\tan^{-1}(\omega RC) \\ A &= \frac{V_0}{\sqrt{(RC\omega)^2 + 1}} \end{aligned} \tag{70}$$

The initial condition c is derived at $V_o(0) = 0$, since the response of an passive circuit is non-instant.

$$\begin{aligned} V_o(0) = 0 &= ce^{\frac{0}{RC}} + A\cos(\omega 0 + \Phi) \\ 0 &= c + A\cos(\Phi) \\ c &= -A\cos(\Phi) \end{aligned} \tag{71}$$

Since this circuit is very common, we can verify it against literature to ensure that our equation is correct. [21]

Our general solution generates the output function given input function along with the circuit's resistance and capacitance.

We can graph $V_i(t)$ and $V_o(t)$ on the same axes with Desmos [22] to visualize the behaviour of the circuit.

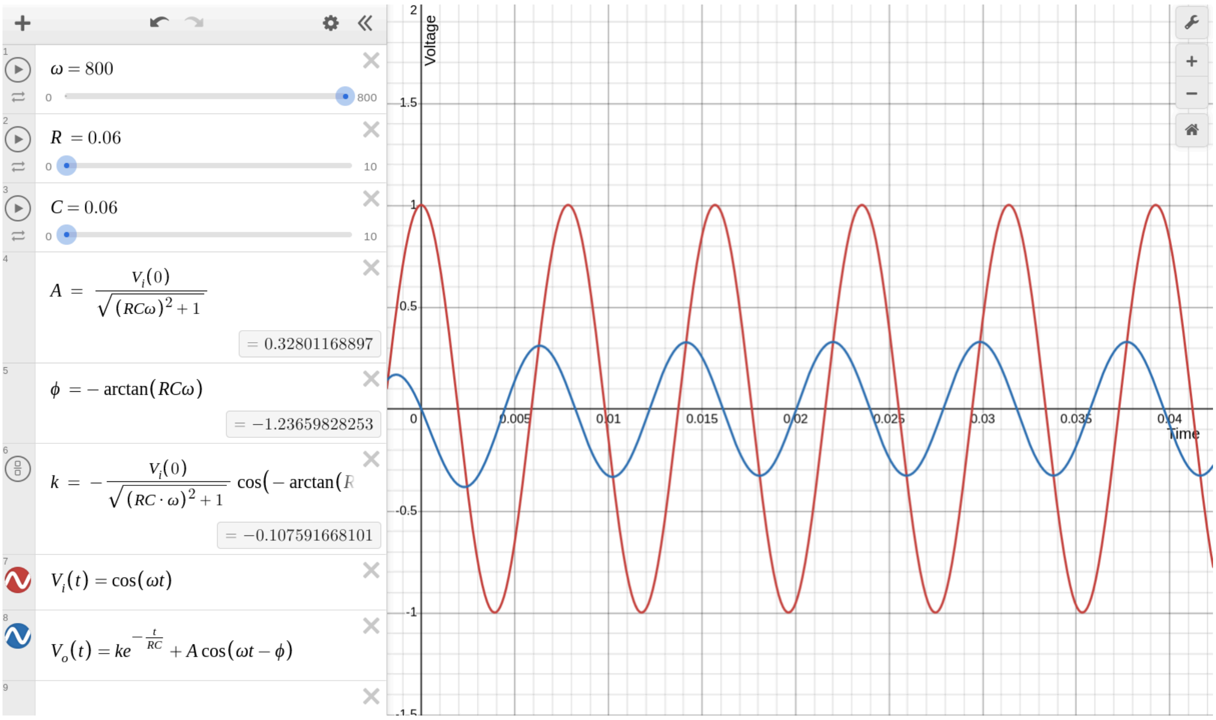


Figure 7: Low-pass filter at 800Hz

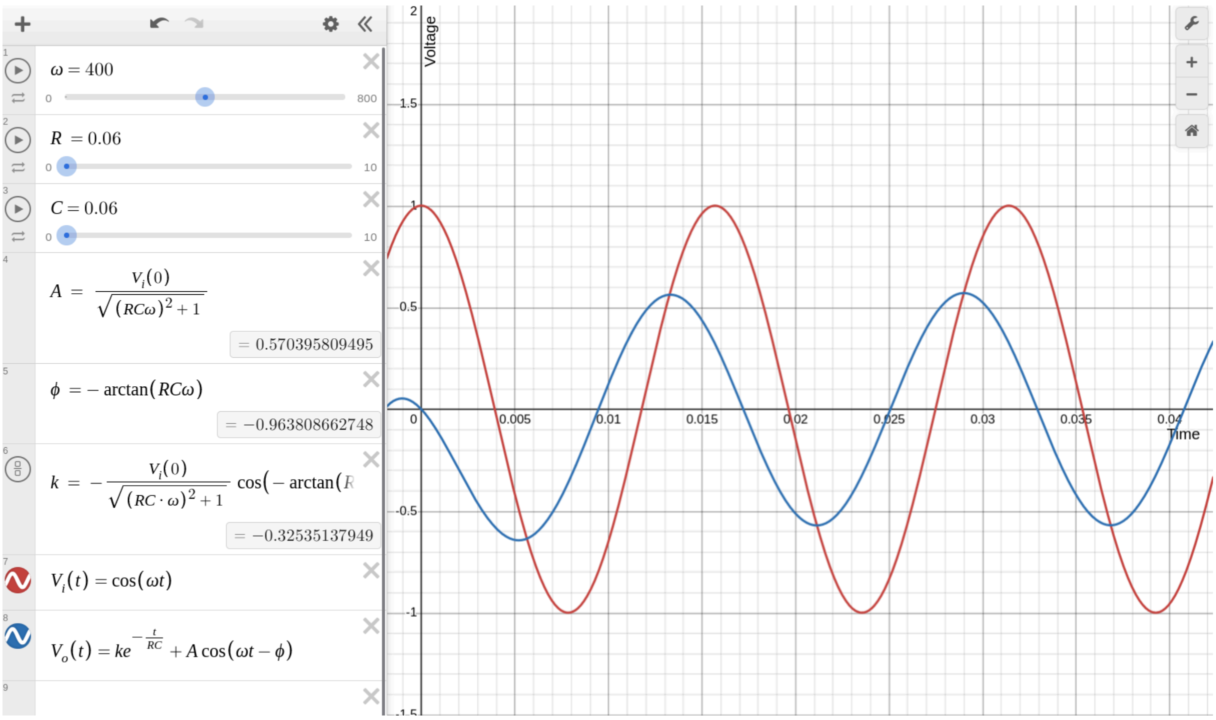


Figure 8: Low-pass filter at 400Hz

Find the solution at https://www.desmos.com/calculator/8c1l1lh5wz°

Comparing these two graphs of the same circuit, it is clear at 800Hz input voltage, the amplitude of the output is smaller relative to the output amplitude of 600Hz of input voltage.

This means our model of the low-pass filter is working as expected. However, this solution is limited as it can only describe the transformation of a certain input voltage.

To be able to better understand the filter, we need to understand how the filter transforms amplitude in accordance to a range of frequencies.

To achieve this, we will utilize the Laplace Transform.

5.2.4. The Laplace Transform

The Laplace Transform (LT) is an integral-valued transform that converts a function in the time domain ($v(t)$) into a function of the complex frequency domain, also known as the s-domain. [23]

It is defined as:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t = F(s) \tag{72}$$

Where $s = x + iy$ is the complex frequency variable, allowing for us to solve independent of time.

This is perfect for us, as Euler's form for sinusoidals makes use of complex valued frequency in terms of $j\omega$, and we have already described the behaviour of our components alternatively with $j\omega$. So, for all of our future LTs, we will take $s = j\omega$. [24]

Since we have already described the behaviour of RLC components using complex frequency, we can immediately convert them into the S-domain.

Component	Time Domain	S-Domain
Resistor	$i \cdot R$	$i \cdot R$
Inductor	$\frac{\mathrm{d}i}{\mathrm{d}t} L$	$i \cdot j\omega L$
Capacitor	$C \int i\,\mathrm{d}t$	$i \cdot \frac{1}{j\omega C}$

Note how the Laplace transform reduces calculus operations into just algebra, making it a lot easier for us to work with ODEs.

To be able to compare the frequencies, we define a transfer function, $H(s)$.

$$H(s) = \frac{V_o(s)}{V_i(s)} \tag{73}$$

Since we have already defined both $V_i(t)$ and $V_o(t)$ in terms of voltage drops, we can convert them into the S-domain easily.

$$V_o(s) = i \cdot \frac{1}{j\omega C} \tag{74}$$

$$\begin{aligned} V_i(s) &= i \cdot R + V_o(s) \\ &= i \cdot R + i \cdot \frac{1}{j\omega C} \\ &= i\left(R + \frac{1}{j\omega C}\right) \end{aligned} \tag{75}$$

Hence, the transfer function will be

$$\begin{aligned} H(s) &= \frac{i \cdot \frac{1}{j\omega C}}{i\left(R + \frac{1}{j\omega C}\right)} \\ &= \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} \\ &= \frac{1}{j\omega RC + 1} \end{aligned} \tag{76}$$

Now that we have the transfer function, we can draw a bode plot. A Bode plots frequency as the horizontal axis.

We can further plot the magnitude of the transfer function to compare the change in voltage amplitude over a range of frequency

$$|H(j\omega)| = \frac{1}{\sqrt{1 + (\omega RC)^2}} \tag{77}$$

However, the magnitude change can be quite large, so we will convert it from a voltage ratio to decibels, which uses a logarithmic scale. [25]

$$|H_{\text{db}}(j\omega)| = -20 \log \sqrt{1 + (\omega RC)^2} \tag{78}$$

Due to the large frequency scale, we will use a logarithmic scale for it.

We can also Bode plot the phase change with. [26]

$$\angle H(j\omega) = -\arctan(\omega RC) \tag{79}$$

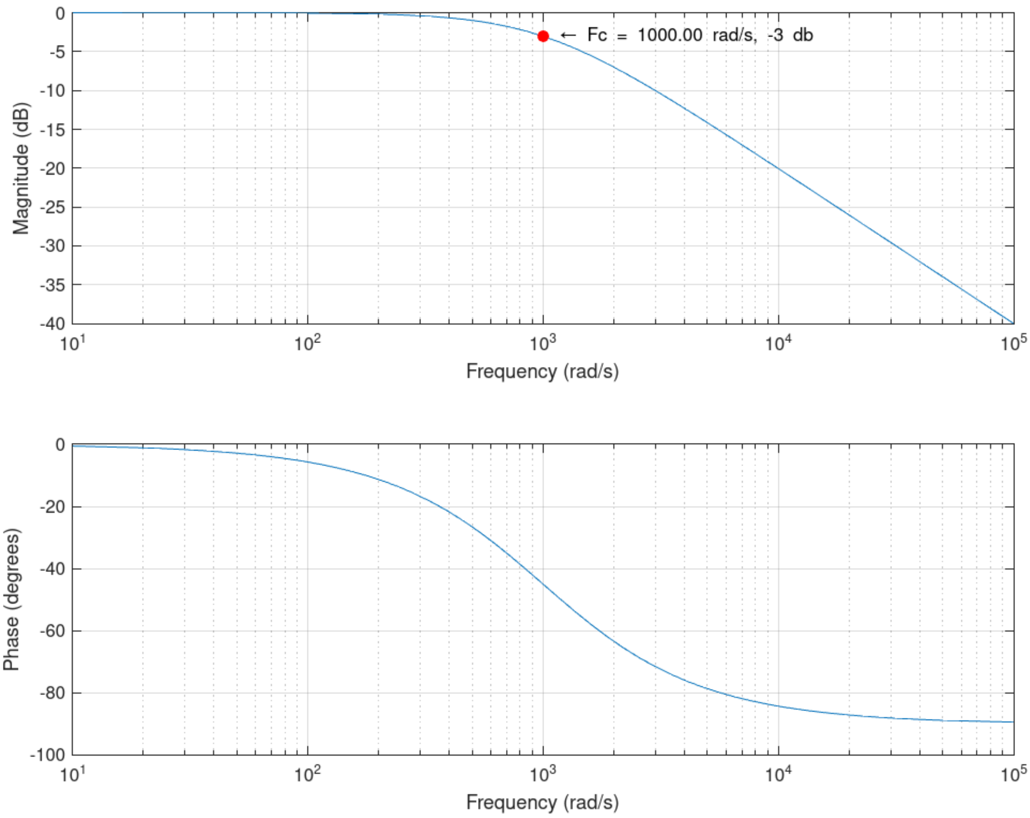


Figure 9: Magnitude and phase bode plot of $H(s)$

As we can see, the magnitude of the transfer function shallowly rolls off for high frequencies.

Notably, the magnitude rolls off specifically to $-3dB$ or $1000\frac{\text{rad}}{s}$

The frequency at which the magnitude is $-3dB$ is described as the cut-off frequency, this is the value that is manipulated for low-pass filters. [27]

This gain of $-3dB$ can also be written as a change in voltage

$$\begin{aligned} \text{Gain (dB)} &= 20\log_{10}(|H(j\omega)|) \\ -3dB &= 20\log_{10}(|H(j\omega)|) \\ -0.15 &= \log_{10}(|H(j\omega)|) \\ |H(j\omega)| &= 10^{-0.15} \\ |H(j\omega)| &\approx \frac{1}{\sqrt{2}} \end{aligned} \tag{80}$$

$$\begin{aligned} |H(j\omega)| = \frac{1}{\sqrt{2}} &= \frac{1}{\sqrt{1+(\omega RC)^2}} \\ 2 &= 1+(\omega RC)^2 \\ 1 &= (\omega RC)^2 \\ \omega &= \frac{1}{RC} \end{aligned} \tag{81}$$

Since angular frequency, $\omega = 2\pi f$ where f is frequency.

$$\begin{aligned} 2\pi f_c &= \frac{1}{RC} \\ f_c &= \frac{1}{2\pi RC} \end{aligned} \tag{82}$$

We have derived this equation, which relates the cut-off frequency for a first-order RC low-pass filter to its component values.

6. Conclusion

This extended essay explored the applications and limitations of ordinary differential equations (ODEs) in modelling and analyzing the behavior of electrical filters, with a particular focus on low-pass filters. The exploration began with a foundational understanding of electrical circuits, emphasizing the behaviour of resistors, inductors, and capacitors. These components form the backbone of filter design, each contributing unique characteristics that influence the overall response to electrical signals.

We further discussed mathematical modelling of these circuits using ODEs. By employing both numerical and analytical methods, we established a clear framework for predicting how filters respond to various inputs. Numerical methods, such as Euler's method, offer approximate solutions that are invaluable when dealing with complex, non-linear systems where analytical solutions are impractical and incredibly complex. On the other hand, analytical solutions provide precise insights into system dynamics, allowing for the fine-tuning of parameters to achieve desired outcomes.

However, this abstract model is limited. The equations for the component behaviours were assumed to be axiomatic, but in real life, full efficiency is impossible for any electrical circuit. Furthermore, the behaviour equations for the components assume ideal behaviour, so there will be discrepancies between experimental testing and theoretical derivation.

Although this extended essay realized its goal, to further this investigation, we could look at more complex filters, such as the band-pass filter. This filter requires a second-order ordinary differential equation to model and, thus, would have a more complex solution. We could also look at the similarity between modeling electrical circuits and a spring-mass damping system, as they have the same behavioural characteristics. Lastly, we could take into account component inefficiencies to make our model more precise and applicable in the real world.

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